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GENERICITY RESULTS FOR SINGULAR CURVES

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Abstract

Let M be a smooth manifold and $\mathcal{D}_m$, $m \geqslant 2$, be the set of rank m distributions on M endowed with the Whitney C^∞ topology. We show the existence of an open set O_m dense in $\mathcal{D}_m$, so that every nontrivial singular curve of a distribution D of O_m is of minimal order and of corank one. In particular, for $m \geqslant 3$, every distribution of O_m does not admit nontrivial rigid curves. As a consequence, for generic sub-Riemannian structures of rank greater than or equal to three, there do not exist nontrivial minimizing singular curves.

1. Introduction

Let M be a smooth paracompact manifold of dimension n, and D be a *distribution* on M that is a subbundle of the tangent bundle TM of M. All vector spaces $D(q)$, $q \in M$, have dimension $m \leqslant n$, called the *rank* of the distribution D. A curve $q(\cdot) : [0,1] \to M$ is said to be *horizontal* if it is absolutely continuous and $\dot{q}(t) \in D(q(t))$, for almost every $t \in [0,1]$.

For $q_0 \in M$, let $\Omega(q_0)$ be the set of horizontal curves $q(\cdot) : [0,1] \to M$ such that $q(0) = q_0$. The set $\Omega(q_0)$, endowed with the $W^{1,1}$-topology, inherits of a Banach manifold structure[1] .

For $q_0, q_1 \in M$, let $\Omega(q_0, q_1)$ be the set of horizontal curves $q(\cdot) : [0,1] \to M$ such that $q(0) = q_0$ and $q(1) = q_1$. Notice that $\Omega(q_0, q_1) = \mathrm{End}_{q_0}^{-1}(q_1)$, where the end-point mapping $\mathrm{End}_{q_0} : \Omega(q_0) \to M$ is the smooth mapping defined by $\mathrm{End}_{q_0}(q(\cdot)) = q(1)$.

Definition. A curve $q(\cdot)$ is said to be *singular* if it is horizontal and if it is a critical point of the end-point mapping $\mathrm{End}_{q(0)}$. The codimension of the singularity is called the *corank* of the singular curve.

The set $\Omega(q_0, q_1)$ is a Banach submanifold of $\Omega(q_0)$ of codimension n in a neighborhood of a nonsingular curve, but may fail to be a manifold in a neighborhood of a singular curve. Hence the study of singular curves is of crucial importance in the calculus of variations with nonholonomic

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[1]It is a straightforward adaptation of results of Bismut [**9**].

constraints. Recently, the geometry of the set $\Omega(q_0, q_1)$ has received a revival of interest, for instance in Griffiths [**19**], Hamenstädt [**20**], Pansu [**29**], Strichartz [**32**], or Zhong Ge [**35**].

Singular curves first appeared in the works of Carathéodory, Engel, and Hilbert (see [**10**, **34**]), through the notion of *rigidity*. Recall that rigid curves are locally isolated curves in $\Omega(q_0, q_1)$ for the $W^{1,\infty}$-topology, and form a particular class of singular curves. More recently, Bryant and Hsu prove in [**13**] that every rank two distribution satisfying some mild non-degeneracy conditions possesses rigid curves. In [**5**], Agrachev and Sarychev develop a second-order variation theory in order to characterize rigid curves.

In the theory of classification of distributions, singular curves are natural candidates to be invariant geometric objects. The question is to know whether or not a distribution is characterized, up to diffeomorphism, by the set of its singular curves. The answer is in general negative, due to the existence of moduli of normal forms. However, the answer remains positive for a large class of distributions, see Jakubczyk and Zhitomirskii [**24**], and Montgomery [**27**].

Singular curves play a major role in the framework of sub-Riemannian geometry (also known as Carnot-Carathéodory geometry). Recall that every sub-Riemannian minimizing curve is either a singular curve, or the projection of a normal extremal, i.e., a solution of the geodesic equations associated to the sub-Riemannian metric. Note that singular curves do not depend on the metric, but may be minimizing. Attempts have been made, however, to ignore singular curves, on the false grounds that they are never optimal. In [**26**], Montgomery offers both an example of a minimizing singular curve, which is not the projection of a normal extremal, and a list of false proofs (by several authors) allegedly showing that a singular curve cannot be optimal. These findings gave impetus to wide-ranging research with view to identifying the role of singular curves in sub-Riemannian geometry, and in particular their optimality status (see for instance Agrachev and Sarychev [**4**], Liu and Sussmann [**25**]). Besides, the existence of minimizing singular curves is closely related to the regularity of the sub-Riemannian distance in the analytic context. In [**1**], the author proves that, in the absence of a nontrivial minimizing singular curve starting from q_0, the sub-Riemannian distance $d_{SR}(q_0, \cdot)$ is subanalytic outside q_0. In [**2**], Agrachev and Gauthier show that this situation is valid for a dense set (for the Whitney topology) of distributions of rank greater than or equal to three.

The existence of singular curves has consequences in the theory of hypoelliptic operators; singular curves have an impact on the asymptotics of the spectrum of a certain class of sub-Laplacian operators whose symbols correspond to sub-Riemannian metrics, see [**27**]. This fact seems to be general (see [**9**, **17**]) but, up to now, has not been completely cleared

up. Christ has conjectured that, in presence of singular curves, hypoelliptic sub-Laplacian operators may fail to be analytic hypoelliptic (see [**15**, **16**]). This is related to a conjecture of Treves [**33**].

In this paper, we first give two geometric characterizations of singular curves in terms of characteristic curves. The first one was discovered by Hsu [**23**]; the second one (Proposition 2.3) is new, and is the starting point of our analysis. Indeed, this proposition puts forward a relevant property of singular curves, namely to be of *minimal order*. Roughly speaking, minimal order means that a minimal amount of time differentiations is sufficient to recover the field of characteristic directions defining the singular curve. This terminology was introduced by Bonnard and Kupka [**12**] in the context of control theory. Our main result, Theorem 2.4, states that, for generic distributions, nontrivial singular curves are of minimal order and of corank one. Here, a distribution is said to be generic if it belongs to an open dense subset of the set of distributions endowed with the Whitney topology. This result has several consequences. First, it implies that generic distributions of rank greater than or equal to three do not admit rigid curves (Theorem 2.6). This answers positively to a conjecture of Bryant and Hsu [**13**]. Second, for generic sub-Riemannian geometry structures of rank greater than or equal to three, there do not exist nontrivial minimizing singular curves (Theorem 2.8). We thus extend results of [**25**], and also improve some results of [**2**].

Some results of the present paper were announced in [**14**].

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2. Singular curves of distributions

2.1. Characterizations of singular curves. Let T^*M denote the cotangent bundle of M, $\pi : T^*M \to M$ the canonical projection, and ω the canonical symplectic form on T^*M. We use $D^\perp$ to denote the annihilator of D in T^*M minus its zero section. We define $\overline{\omega}$ to be the restriction of ω to $D^\perp$; this restriction needs not be symplectic, and hence it might admit characteristic subspaces $\ker \overline{\omega}(\psi)$ at $\psi \in D^\perp$.

Definition. An absolutely continuous curve $\psi(\cdot) : [0,1] \to D^\perp$ such that $\dot{\psi}(t) \in \ker \overline{\omega}(\psi(t))$ for almost every $t \in [0,1]$, is called an *abnormal extremal* of D.

Such a curve is sometimes called a *characteristic curve* of $\overline{\omega}$. Here, we adopt the terminology stemming from calculus of variations. The result of [**23**] given next (see also [**30**]) provides a first characterization of singular curves.

Proposition 2.1. *A curve $q(\cdot) : [0,1] \to M$ is singular if and only if it is the projection of an abnormal extremal $\psi(\cdot)$ of D. The curve $\psi(\cdot)$ is said to be an abnormal extremal lift of $q(\cdot)$.*

Remark 1. The set of abnormal extremals lifts of a given singular curve $q(\cdot)$ is a vector space whose dimension is the corank of $q(\cdot)$. In particular, when $q(\cdot)$ is of corank one, it admits a unique (up to a scalar) abnormal extremal lift.

Remark 2. Every constant curve is singular if $m < n$. For the rest of the paper, a curve not reduced to a point is said to be *nontrivial*. If $m = n$, then $D = TM$ and there is no singular curve.

We next provide a Hamiltonian characterization of singular curves.

For a smooth function h on T^*M, we denote by $\overrightarrow{h}$ the Hamiltonian vector field on T^*M defined by $i_{\overrightarrow{h}}\omega = -dh$. Given a smooth vector field f on M, we denote by h_f the function on T^*M defined by $h_f(\psi) = \psi(f)$.

For every $\psi \in D^\perp$, we define $\overrightarrow{h_D}(\psi)$ as the subset of $T_\psi(T^*M)$ of all elements $\overrightarrow{h_f}(\psi)$, where f is a smooth section of D. Notice that for every smooth function α on M, one has

$$\overrightarrow{h_{\alpha f}}(\psi) = \alpha(\pi(\psi))\overrightarrow{h_f}(\psi)$$

for every $\psi \in D^\perp$. Hence $\overrightarrow{h_D}$ is a rank m subbundle of $T(T^*M)$ with basis $D^\perp$.

Remark 3. Notice that $\overrightarrow{h_D} = \mathrm{orth}_\omega(TD^\perp)$, where orth_ω denotes the symplectic orthogonal with respect to ω. Indeed, for every $\psi \in D^\perp$, there holds

$$\begin{aligned} T_\psi D^\perp &= \{dh_f(\psi) = 0 \ : \ f \in D\} \\ &= \{\xi \in T_\psi(T^*M) \ : \ \omega(\overrightarrow{h_f}, \xi) = 0, \ \forall f \in D\} \\ &= \mathrm{orth}_\omega(\overrightarrow{h_D}(\psi)). \end{aligned}$$

Definition. We define ω_D as the restriction of ω to the subbundle $\overrightarrow{h_D}$, that is

$$\omega_D(\psi) = \omega(\psi)_{|\overrightarrow{h_D}(\psi)},$$

for every $\psi \in D^\perp$. Equivalently, if $j : \overrightarrow{h_D} \hookrightarrow T(T^*M)$ denotes the canonical injection, one has $\omega_D = j^*\omega$.

It follows readily from Remark 3 that every abnormal extremal $\psi(\cdot)$ of D satisfies, for a.e. $t \in [0,1]$, $\dot\psi(t) \in \overrightarrow{h_D}(\psi(t))$ and $\omega_D(\psi(t))(\dot\psi(t), \cdot) = 0$. As a consequence, the rank of $\omega_D(\psi(t))$ is less than m for every $t \in [0,1]$.

If, moreover, m is even, then $\omega_D^{m/2}(\psi(\cdot)) \equiv 0$. In order to differentiate this relation, we need the following lemma.

Lemma 2.2. *Let $\psi_0 \in D^\perp$ such that $\omega_D^{m/2}(\psi_0) = 0$. For every $\xi \in \overrightarrow{h_D}$, the m-form $L_\xi \omega_D^{m/2}(\psi_0)$ on $\overrightarrow{h_D}(\psi_0)$, where L_ξ denotes the Lie derivative, only depends on ξ_{ψ_0}, the value of ξ at ψ_0. We use $L_{\xi_{\psi_0}} \omega_D^{m/2}(\psi_0)$ to denote this m-form.*

Proof. It is enough to prove that $L_\xi \omega_D^{m/2}(\psi_0) = 0$ whenever $\xi(\psi_0) = 0$. For such a ξ, one has easily

$$[\xi, \overrightarrow{h_D}](\psi_0) \subset \overrightarrow{h_D}(\psi_0),$$

where $[\ ,\ ]$ denotes the Lie bracket, and thus

$$\begin{aligned} L_\xi \omega_D^{m/2}(\psi_0)(Y_1, \dots, Y_m) &= \xi.\ \omega_D^{m/2}(\psi_0)(Y_1, \dots, Y_m) \\ &\quad - \sum_{i=1}^{m} \omega_D^{m/2}(\psi_0)(Y_1, \dots, [\xi, Y_i], \dots, Y_m) \\ &= 0. \end{aligned}$$

q.e.d.

It is now immediate that, along the abnormal extremal $\psi(\cdot)$, there holds

$$L_{\dot\psi(t)} \omega_D^{m/2}(\psi(t)) = 0 \quad \text{for a.e. } t \in [0,1].$$

This suggests to introduce, for $\psi_0 \in D^\perp$ such that $\omega_D^{m/2}(\psi_0) = 0$, the linear mapping

$$\begin{array}{rcl} \widetilde{\omega}_D(\psi_0) : \overrightarrow{h_D}(\psi_0) & \longrightarrow & \Lambda^1(\overrightarrow{h_D}(\psi_0)) \times \Lambda^m(\overrightarrow{h_D}(\psi_0)) \\ \xi_{\psi_0} & \longmapsto & (\omega_D(\psi_0)(\xi_{\psi_0}, \cdot), L_{\xi_{\psi_0}} \omega_D^{m/2}(\psi_0)), \end{array}$$

where the notation $\Lambda^k(\cdot)$ stands for the set of k-forms on a vector space.

We finally obtain the following characterization for singular curves.

Proposition 2.3. *An absolutely continuous curve $\psi(\cdot) : [0,1] \to D^\perp$ is an abnormal extremal of D if and only if*

- $\dot\psi(t) \in \ker \omega_D(\psi(t))$ *a.e. if m is odd,*
- $\dot\psi(t) \in \ker \widetilde{\omega}_D(\psi(t))$ *a.e. if m is even.*

If m is odd (resp. if m is even), the property $\dim \ker \omega_D(\psi) = 1$ (resp. $\dim \ker \widetilde{\omega}_D(\psi) = 1$) is open in $D^\perp$ (resp. in $\{\omega_D^{m/2} = 0\}$). As a consequence, for every abnormal extremal $\psi(\cdot) : [0,1] \to D^\perp$ of D, if there exists $t_0 \in [0,1]$ such that $\dim \ker \omega_D(\psi(t_0)) = 1$ if m is odd (resp. $\dim \ker \widetilde{\omega}_D(\psi(t_0)) = 1$ if m is even), then it is possible to define, in a neighborhood of $\psi(t_0)$, a unique field of characteristic directions and thus to recover locally the abnormal extremal $\psi(\cdot)$ up to reparametrization.

Moreover, when m is odd, if $\dim \ker \omega_D(\psi(\cdot)) = 1$ a.e. along every abnormal extremal $\psi(\cdot)$ of D, then there exists an open dense subset

of M such that, through every point of this subset, passes a nontrivial singular curve (see also [**27**]).

This motivates the following definition.

Definition. A singular curve $q(\cdot) : [0,1] \to M$ is said to be of *minimal order* if it admits an abnormal extremal lift $\psi(\cdot) : [0,1] \to D^\perp$ such that $\dim \ker \omega_D(\psi(t)) = 1$ a.e. if m is odd, and $\dim \ker \widetilde{\omega}_D(\psi(t)) = 1$ a.e. if m is even.

On the opposite, for arbitrary m, a singular curve is said to be a *Goh curve* if it admits an abnormal extremal lift $\psi(\cdot)$ along which $\omega_D(\psi(\cdot)) \equiv 0$. A Goh curve cannot be of minimal order when $m \geqslant 3$.

Remark 4. Let $q(\cdot)$ be a singular curve. For an abnormal extremal lift $\psi(\cdot)$ of $q(\cdot)$, the function $k_\psi : t \mapsto \dim \ker \omega_D(\psi(t))$ (resp. $t \mapsto \dim \ker \widetilde{\omega}_D(\psi(t))$) needs not be constant a.e., and is only upper semicontinuous in general. Moreover, if the singular curve $q(\cdot)$ is of corank greater than one, it admits several linearly independent abnormal extremal lifts. The functions $k_\psi(\cdot)$ associated to each of these lifts are not related to one another. It is then not obvious in general to define a geometric invariant using the functions $k_\psi(\cdot)$. The only geometric invariant considered here is the corank of $q(\cdot)$ (i.e., the codimension of the singularity of the end-point mapping).

This emphasizes the relevance of the notion of minimal order. As noted above, it permits to recover the field of characteristics. Furthermore, it turns out to be a generic property of singular curves, as shown in the main result hereafter.

2.2. The main result. Let M be a smooth manifold of finite dimension. The following theorem constitutes the main result of the paper.

Theorem 2.4. *Let $m \geqslant 2$ be a positive integer and let $\mathcal{D}_m$ be the set of rank m distributions on M endowed with the Whitney C^∞ topology. There exists an open set O_m dense in $\mathcal{D}_m$ so that every nontrivial singular curve of a distribution D of O_m is of minimal order and of corank one.*

Remark 5. In addition, for every integer k, the set O_m can be chosen so that its complement has codimension greater than k. Let O_m^∞ be the intersection over all k of the latter subsets; then O_m^∞ shares the same properties as the set O_m with the following differences: O_m^∞ may fail to be open, but its complement has infinite codimension.

Corollary 2.5. *If $m \geqslant 3$, then every distribution $D \in O_m$ does not admit nontrivial Goh singular curves.*

Remark 6. In particular, every distribution D of O_m^∞ has no nontrivial Goh singular curve. This is exactly the contents of [**2**, Theorem 8].

Recall that a curve $q(\cdot) \in \Omega(q_0, q_1)$ of a distribution D is *rigid* if it is isolated (up to reparametrization) in $\Omega(q_0, q_1)$ endowed with the $W^{1,\infty}$-topology. A rigid curve has to be a Goh curve (see [**5**]), and hence, we get the following result.

Theorem 2.6. *If $m \geqslant 3$, then every distribution $D \in O_m$ has no nontrivial rigid curve.*

This answers positively to a conjecture of Bryant and Hsu [**13**], who proved the result for generic distributions of rank 3 in dimension 5 or 6.

2.3. Consequences in sub-Riemannian geometry. Recall that a *sub-Riemannian* manifold is a 3-tuple (M, D, g), where M is a smooth manifold of finite dimension, D is a distribution on M and g is a Riemannian metric defined on D. A sub-Riemannian manifold is analytic if M, D, g are.

The *sub-Riemannian distance* $d_{SR}(q_0, q_1)$ between two points q_0, q_1 of M is the infimum over the Riemannian lengths (for the metric g) of the horizontal curves joining q_0 and q_1. Such a horizontal curve is called a *minimizing curve* if its length is equal to $d_{SR}(q_0, q_1)$. The *sub-Riemannian sphere* $S(q_0, r)$ centered at q_0 with radius r is the set of points $q \in M$ such that $d_{SR}(q_0, q) = r$.

Let (M, D, g) be a sub-Riemannian manifold. We define the Hamiltonian $H : T^*M \to \mathbb{R}$ as follows. For every $q \in M$, the restriction of H to the fiber T^*_qM is given by the nonnegative quadratic form

$$\lambda \longmapsto \frac{1}{2} \max \left\{ \frac{\lambda(v)^2}{g_q(v, v)} \ : \ v \in D(q) \setminus \{0\} \right\}. \tag{1}$$

A *normal extremal* is an integral curve of $\overrightarrow{H}$ defined on $[0, 1]$, i.e., a curve $\psi(\cdot) : [0, 1] \to T^*M$ such that $\dot{\psi}(t) = \overrightarrow{H}(\psi(t))$, for $t \in [0, 1]$. Notice that the projection of a normal extremal is a horizontal curve.

According to the Pontryagin maximum principle (see [**30**]), a necessary condition for a curve to be minimizing is to be the projection either of a normal extremal or of an abnormal extremal. In particular, singular curves satisfy this condition. However, a singular curve may also be the projection of a normal extremal.

Definition. A singular curve is said to be *strictly abnormal* if it is not the projection of a normal extremal.

Remark 7. A singular curve is of corank one if it admits a unique (up to a scalar) abnormal extremal lift. It is strictly abnormal and of corank one if it admits a unique (up to a scalar) extremal lift which is abnormal.

Let M be a smooth manifold. The next result is proved in the preprint [**11**]. For the sake of completeness, a proof of that result is given in the appendix.

Proposition 2.7. *Let $m \geqslant 2$ be a positive integer, $\mathcal{G}_m$ be the set of couples (D, g), where D is a rank m distribution and g is a Riemannian metric on D, endowed with the Whitney C^∞ topology. Then there exists an open dense subset W_m^s of $\mathcal{G}_m$ such that every nontrivial singular curve of an element of W_m^s is strictly abnormal.*

According to [**6**, Theorem 3.7], a minimizing singular curve which is strictly abnormal is necessarily a Goh curve[2] . Hence, combining the above proposition and Corollary 2.5, we get the next result.

Theorem 2.8. *Let $m \geqslant 3$ be a positive integer. There exists an open dense subset W_m of $\mathcal{G}_m$ such that every element of W_m does not admit nontrivial minimizing singular curves.*

Remark 8. In addition, for every integer k, the sets W_m and W_m^s can be chosen so that their complements have codimension greater than k. As in Remark 5, we obtain a subset W_m^∞ of $\mathcal{G}_m$, sharing the same properties as W_m, which may fail to be open but whose complement has infinite codimension.

The absence of nontrivial minimizing singular curves has consequences on the regularity of the sub-Riemannian distance d_{SR}. More precisely, in an analytic context, if there is no nontrivial minimizing singular curve in $\Omega(q)$, then $d_{SR}(q, .)$ is subanalytic in a pointed neighborhood of q in M, and thus the sub-Riemannian spheres $S(q, r)$ with small positive radius r are subanalytic (see [**1**]). For a general definition of subanalyticity, see e.g., [**21**, **22**]. The next result then follows from Theorem 2.8.

Corollary 2.9. *Assume that M is an analytic manifold and, for $m \geqslant 3$, let $\mathcal{G}_m^\omega$ be the set of analytic couples (D, g) on M endowed with the Whitney topology. Then, there exists an open dense set W_m of $\mathcal{G}_m^\omega$ so that, for every element $(D, g) \in W_m$, the sub-Riemannian spheres $S(q, r)$ with small positive radii are subanalytic.*

Remark 9. As in Remarks 5, 6 and 8, we obtain a subset W_m^∞ of $\mathcal{G}_m^\omega$, sharing the same properties as W_m, which may fail to be open but whose complement has infinite codimension. We thus recover [**2**, Theorem 9].

Remark 10. Corollaries 2.5 and 2.9 are the only results of the present paper similar to results of [**2**]. Both are actually stronger than [**2**, Theorems 8 and 9], in which only the existence of a subset of infinite codimension is proved (see Remarks 6 and 9). The difference is the openness property, stated in Corollary 2.5 (and then in Corollary 2.9), which is essential to derive the conclusion of Theorem 2.8.

[2]For a more detailed and self-contained proof of that result, see the textbook [**3**] and, more specifically, Theorem 20.6 page 300 and Proposition 20.13 page 314 therein.

2.4. Formulation in local coordinates. In this section, we translate into local coordinates the objects introduced in Section 2.1.

For every open set $U \subset M$, let $VF(U)$ be the set of smooth vector fields on U. We use $VF^m(U)$ (resp. $VF_0^m(U)$) to denote the set of m-tuples of elements (resp. everywhere linearly independent) of $VF(U)$ and we use $D_{|U}$ to denote the restriction of the distribution D to U. Let $q_0 \in M$, and U be an open neighborhood of q_0 in M such that $D_{|U}$ is spanned by a m-tuple $(f_1, \dots, f_m) \in VF_0^m(U)$.

Every curve $q(\cdot) \in \Omega(q_0)$, contained in U, satisfies

$$\dot{q}(t) = \sum_{i=1}^{m} u_i(t) f_i(q(t)) \text{ for a.e. } t \in [0,1],$$

where $u_i \in L^1([0,1], \mathbb{R})$, for $i = 1, \dots, m$. The function $u(\cdot)$ defined by $u(\cdot) = (u_1(\cdot), \dots, u_m(\cdot))$ is called the *control* associated to $q(\cdot)$.

For $i, j \in \{1, \dots, m\}$, set $h_i = h_{f_i}$ and $h_{ij} = h_{[f_i, f_j]}$. Notice that $h_{ij} = \{h_i, h_j\}$, where $\{\cdot, \cdot\}$ denotes the Poisson bracket. The vector fields $\overrightarrow{h_1}, \dots, \overrightarrow{h_m}$ form a field of frame of the restriction of $\overrightarrow{h_D}$ to $D_{|U}^{\perp}$. In the coordinates defined by this field of frame, the form $\omega_D(\psi)$ is represented by the skew-symmetric $(m \times m)$-matrix

$$G(\psi) = \big(h_{ij}(\psi)\big)_{1 \leqslant i,j \leqslant m}, \tag{2}$$

for $\psi \in D_{|U}^{\perp}$. We call G the *Goh matrix* associated to the field of frame $(f_1, \dots, f_m)$.

Let Vol denote the volume form, in the previous coordinates, on the restriction of $\overrightarrow{h_D}$ to $D_{|U}^{\perp}$. When m is even, the m-form $\omega_D^{m/2}$ is equal to $P\,\mathrm{Vol}$, where $P : D_{|U}^{\perp} \to \mathbb{R}$ denotes the *Pfaffian* of the Goh matrix G, i.e., $P(\psi)^2 = \det G(\psi)$, for $\psi \in D_{|U}^{\perp}$. The Pfaffian P is a homogeneous polynomial of degree $m/2$ in the h_{ij}, see [**7**]. In local coordinates, $\widetilde{\omega}_D(\psi)$ is represented by the $((m+1) \times m)$-matrix $\widetilde{G}(\psi)$, defined as $G(\psi)$ augmented with the row $(\{P, h_j\}(\psi))_{1 \leqslant j \leqslant m}$, for $\psi \in D_{|U}^{\perp}$.

Let $q(\cdot) \in \Omega(q_0)$ be a singular curve contained in U. It is the projection of an abnormal extremal $\psi(\cdot)$. In the local coordinates, $\psi(t) \in D^{\perp}$ means that, for $t \in [0,1]$,

$$h_i(\psi(t)) = 0, \quad i = 1, \dots, m. \tag{3}$$

Moreover, since $\dot{\psi}(t) \in \overrightarrow{h_D}(\psi(t))$ for a.e. $t \in [0,1]$, we have

$$\dot{\psi}(t) = \sum_{i=1}^{m} u_i(t) \overrightarrow{h_i}(\psi(t)), \tag{4}$$

where $u(\cdot) = (u_1(\cdot), \dots, u_m(\cdot))$ is the control associated to $q(\cdot)$. From Proposition 2.3, there holds, for a.e. $t \in [0,1]$:

(i) $u(t) \in \ker G(\psi(t))$ if m is odd,

(ii) $u(t) \in \ker \widetilde{G}(\psi(t))$ if m is even.

With the notations above, if m is odd (resp. even), a singular curve is of minimal order if it admits an abnormal extremal lift along which $\operatorname{rank} G(\psi(t)) = m-1$ (resp. $\operatorname{rank} \widetilde{G}(\psi(t)) = m-1$) a.e. on $[0,1]$. It is a Goh curve if $h_{ij}(\psi(\cdot)) \equiv 0$, for $i,j \in \{1,\dots,m\}$.

Remark 11. Differentiating (3) and using (4) yields, for a.e. $t \in [0,1]$,

$$\sum_{j=1}^{m} h_{ij}(\psi(t))u_j(t) = 0, \ \ i = 1,\dots,m. \tag{5}$$

If moreover m is even, the determinant of $G(\psi(\cdot))$, and thus the Pfaffian $P(\psi(\cdot))$, are identically equal to zero on $[0,1]$. After differentiation, one gets, for a.e. $t \in [0,1]$,

$$\sum_{j=1}^{m} \{P, h_j\}(\psi(t))u_j(t) = 0. \tag{6}$$

We recover in this way the characterization (i)-(ii) of the control associated to a singular curve.

Remark 12. In the context of sub-Riemannian geometry, the Hamiltonian H defined by (1) writes locally

$$H(\psi) = \frac{1}{2}\sum_{i=1}^{m} h_i^2(\psi),$$

for $\psi \in T^*U$, provided that $(f_1,\dots,f_m)$ is orthonormal with respect to the associated Riemanian metric g. The control $u(\cdot) = (u_1(\cdot),\dots,u_m(\cdot))$ associated to the projection of a normal extremal is then given by $u_i(t) = h_i(\psi(t))$, $i \in \{1,\dots,m\}$ and $t \in [0,1]$.

2.5. Structure of the proofs. The rest of the paper is organized as follows. Section 3 is devoted to prove the genericity of the minimal order property (see Proposition 3.1), and Section 4 the corank one property (see Proposition 4.1). Theorem 2.4 follows from Propositions 3.1 and 4.1. Finally, Proposition 2.7 is proved in Appendix.

The proofs of these propositions use two kinds of arguments. The first ones consist of transversality techniques. They are inspired by [**12**], as well as the general strategy of the proofs. The second kind of arguments amount to deriving an infinite number of relations in the cotangent bundle and, from them, to extracting enough independent ones.

More precisely, the aim of Section 3 is to construct a set $O'_m \subset \mathcal{D}_m$ sharing all required properties. The set O'_m is first defined locally as the

complement of a bad set (Section 3.1). We next compute the codimension of the typical fiber of the bad set (Lemma 3.7), and, using transversality arguments, we prove that O'_m is open dense (Lemma 3.2). Then, we prove in Section 3.3 that the minimal order property holds locally in O'_m. We finally prove Proposition 3.1 in Section 3.4.

The proofs of the genericity of the corank-one property (Proposition 4.1 in Section 4) and of the strictly abnormal property (Proposition 2.7 in Appendix) follow the same lines.

3. Genericity of the minimal order property

We assume that $2 \leqslant m < n$.

Proposition 3.1. *There exists $O'_m \subset \mathcal{D}_m$, containing an open dense subset of $\mathcal{D}_m$, such that, along every nontrivial abnormal extremal $\psi(\cdot)$ of a distribution D in O'_m, there holds* $\operatorname{rk} \omega_D(\psi(t)) = m-1$ *if m is odd* (*resp.* $\operatorname{rk} \tilde{\omega}_D(\psi(t)) = m-1$ *if m is even*), *for a.e. $t \in [0,1]$.*

In the sequel, we adopt the following notations. For an open subset U of M, define

- JTU: the space of jets of elements of $VF(U)$;
- $J^N TU$, $N \in \mathbb{N}$: the space of N-jets;
- $J^N_m TU$: the fiber product on U defined by $J^N TU \times_U \cdots \times_U J^N TU$;
- $J^N_{m,q}$: the fiber of $J^N_m TU$ at $q \in U$.

Recall that $VF(U)$, $VF^m(U)$ and $VF^m_0(U)$ are, respectively, the set of smooth vector fields on U, the set of m-tuples of elements of $VF(U)$ and the set of m-tuples of everywhere linearly independent elements of $VF(U)$.

The spaces JTU, $J^N TU$, $VF^m(U)$, and $VF^m_0(U)$, are endowed with the Whitney C^∞ topology.

For $k \in \mathbb{N}$, let $I = i_1 \cdots i_k$ be a multi-index of $\{1, \dots, m\}$. The length of I is $|I| = k$. A multi-index $I = ji \cdots i$ with k consecutive occurrences of the index i is denoted as $I = ji^k$. For $F \in VF^m(U)$, $U \subset M$ open set, f_I is the vector field defined by

$$f_I = [[\dots[f_{i_1}, f_{i_2}], \dots], f_{i_k}].$$

We use h_I to denote h_{f_I}, where $h_{f_I}(\psi) = \psi(f_I)$, for $\psi \in T^*U$. Clearly,

$$\{h_I, h_i\} = h_J, \tag{7}$$

where the multi-index J is equal to the concatenation Ii.

3.1. Construction of O'_m.

3.1.1. Elementary determinants. Let U be an open subset of M, and $F \in VF^m(U)$. We use $\mathfrak{S}_m$ to denote the set of permutations with m elements. We introduce next real valued functions on $\mathfrak{S}_m \times T^*U$, that we call *elementary determinants*, and that are defined inductively. Fix $\sigma \in \mathfrak{S}_m$ and $\psi \in T^*U$. For the sake of simplicity, in this section, the index i stands for $\sigma(i)$, and the argument (σ, ψ) in the subsequent matrices and in the elementary determinants is omitted.

Let $r < m$ be an integer. Set

$$G = (h_{ij})_{1\leqslant i,j\leqslant m},\ G^r = (h_{ij})_{1\leqslant i,j\leqslant r},\ \text{and}\ \Delta_0^r = \det G^r, \tag{8}$$

with $\Delta_0^0 = 1$. We next define inductively the following elementary determinants:

- for $k \in \{r+1, \dots, m\}$ and $s \geqslant 0$, (with the convention that the index $m+1$ stands for $r+1$),

$$\Delta_{0,s+1}^{r,k} = \det \left(\begin{array}{c|c} G^r & \big(h_{ik}\big)_{1\leqslant i\leqslant r} \\ \hline \big(\{\Delta_{0,s}^{r,k}, h_j\}\big)_{j=1,\dots,r,k} & \end{array} \right),$$

$$\Delta_{0,0}^{r,k} = \det \left(\begin{array}{c|c} G^r & \big(h_{ik}\big)_{1\leqslant i\leqslant r} \\ \hline \big(h_{(k+1)j}\big)_{1\leqslant j\leqslant r} & h_{(k+1)k} \end{array} \right);$$

- for $p \in \{1, \dots, m-r-1\}$, $k \in \{r+p+1, \dots, m\}$, $s_1, \dots, s_p \geqslant 1$ and $s \geqslant 0$,

$$\Delta_{0,s_1,\dots,s_p,s+1}^{r,r+1,\dots,r+p,k} = \det \begin{pmatrix} \big(h_{ij}\big)_{\substack{1\leqslant i\leqslant r \\ j=1,\dots,r+p,k}} \\ \hline \big(\{\Delta_{0,s_1-1}^{r,r+1}, h_j\}\big)_{j=1,\dots,r+p,k} \\ \vdots \\ \big(\{\Delta_{0,s_1,\dots,s_p-1}^{r,r+1,\dots,r+p}, h_j\}\big)_{j=1,\dots,r+p,k} \\ \big(\{\Delta_{0,s_1,\dots,s_p,s}^{r,r+1,\dots,r+p,k}, h_j\}\big)_{j=1,\dots,r+p,k} \end{pmatrix},$$

and

$$\Delta_{0,s_1,\dots,s_p,0}^{r,r+1,\dots,r+p,k} = \Delta_{0,s_1,\dots,s_p-1}^{r,r+1,\dots,r+p-1,k}.$$

When m is even, as noticed in Section 2.4, the elementary determinants Δ_0^r defined in (8) are squares of polynomials in h_{ij} (for appropriate sets of indices (i,j)), called Pfaffians. Of special interest are the Pfaffians P^{m-2} and P, associated, respectively, to the matrices G^{m-2} and G. We define inductively additional elementary determinants as follows:

- for $k = m-1$ or m, and $s \geqslant 0$,
$$\delta_{s+1}^k = \det \left(\begin{array}{c|c} G^{m-2} & \big(h_{ik}\big)_{1\leqslant i\leqslant m-2} \\ \hline \big(\{\delta_s^k, h_j\}\big)_{1\leqslant j\leqslant m-2} & \{\delta_s^k, h_k\} \end{array} \right),$$
$$\delta_0^{m-1} = \delta_0^m = P;$$
- for $s_1 \geqslant 1$ and $s \geqslant 0$,
$$\delta_{s_1,s+1} = \det \left(\begin{array}{c} \big(h_{ij}\big)_{\substack{1\leqslant i\leqslant m-2 \\ 1\leqslant j\leqslant m}} \\ \hline \big(\{\delta_{s_1-1}^{m-1}, h_j\}\big)_{1\leqslant j\leqslant m} \\ \big(\{\delta_{s_1,s}, h_j\}\big)_{1\leqslant j\leqslant m} \end{array} \right),$$
$$\delta_{s_1,0} = \delta_{s_1-1}^m.$$

3.1.2. Bad set. Let U be an open subset of M, d an integer, and $N = 2d$. For p integer, let $N_{p,d}$ denote the set of $(p+1)$-tuples $\bar{s} = (0, s_1, \ldots, s_p)$ in $\{0\} \times (\mathbb{N}^*)^p$ with $s_1 + \cdots + s_p < d + p$.

The "bad set" $B(d, U)$ is defined as the canonical projection on $J_m^N TU$ of
$$\widehat{B}(d, U) = \{(j_q^N F, \psi) : q = \pi(\psi),\ \psi \in T^*U,\ j_q^N F \in \widehat{B}^0(d, \psi) \cup \widehat{B}^1(d, \psi)\},$$
where $\widehat{B}^0(d, \psi)$ and $\widehat{B}^1(d, \psi)$ are defined below.

Definition of $\widehat{B}^0(d, \psi)$. If $m = 2$, set $\widehat{B}^0(d, \psi) = \emptyset$. Assume $m \geqslant 3$. For $\psi \in T^*U$ with $\pi(\psi) = q$, $\sigma \in \mathfrak{S}_m$, an even integer $r \leqslant m-3$, and $\bar{s} \in N_{p,d}$, $0 \leqslant p < m$, let $\widehat{B}^0(d, \sigma, r, \bar{s}, \psi)$ be the set of elements $j_q^N F \in J_{m,q}^N$ such that:

1) $f_1(q), \ldots, f_m(q)$ are linearly independent;
2) $\Delta_0^r(\sigma, \psi) \neq 0$;
3) for $i = 0, \ldots, p$,
   (a) $\Delta_{0,s_1,\ldots,s_i}^{r,r+1,\ldots,r+i}(\sigma, \psi) \neq 0$;
   (b) for every $k \in \{r+i, \ldots, m\}$ and $s \in \{1, \ldots, s_i - 1\}$,
   $$\Delta_{0,s_1,\ldots,s_{i-1},s}^{r,r+1,\ldots,r+i-1,k}(\sigma, \psi) = 0;$$
4) for every $k \in \{r+p+1, \ldots, m\}$ and $s \in \{1, \ldots, d+p-(s_1+\cdots+s_p)\}$,
$$\Delta_{0,s_1,\ldots,s_p,s}^{r,r+1,\ldots,r+p,k}(\sigma, \psi) = 0.$$

Define $\widehat{B}^0(d, \psi) \subset J_{m,q}^N$ as the union of the sets $\widehat{B}^0(d, \sigma, r, \bar{s}, \psi)$ with $\sigma \in \mathfrak{S}_m$, $r \leqslant m-3$ even, and $\bar{s} \in N_{p,d}$, $0 \leqslant p < m$.

Definition of $\widehat{B}^1(d, \psi)$. If m is odd, set $\widehat{B}^1(d, \psi) = \emptyset$. Assume that $m \geqslant 2$ is even. For $\sigma \in \mathfrak{S}_m$ and a positive integer $s_1 \leqslant d$, define $\widehat{B}^1(d, \sigma, s_1, \psi)$ as the set of $j_q^N F \in J_{m,q}^N$ such that:

1) $f_1(q), \ldots, f_m(q)$ are linearly independent;
2) $\Delta_0^{m-2}(\sigma, \psi) \neq 0$;

3) a) $\delta_{s_1}^{m-1}(\sigma,\psi)\neq 0$ if $s_1<d$;
   b) for $k\in\{m-1,m\}$ and $s=0,\dots,s_1-1$, $\delta_s^k(\sigma,\psi)=0$;
4) for $s\in\{1,\dots,d-s_1\}$, $\delta_{s_1,s}(\sigma,\psi)=0$.

Let $\widehat{B}^1(d,\psi)\subset J^N_{m,q}$ be the union of the sets $\widehat{B}^1(d,\sigma,s_1,\psi)$ with $\sigma\in\mathfrak{S}_m$ and $s_1\leqslant d$.

3.1.3. Definition of O'_m. Let d be an integer, and $N=2d$. For every U open subset of M, set

$$\mathcal{O}_d(U)=\{F\in VF_0^m(U)\ :\ j^NF\notin B(d,U)\}. \tag{9}$$

Finally, the set O'_m is defined as follows. A distribution D belongs to O'_m if, for every $q\in M$, there exist a neighborhood U of q and a m-tuple of vector fields $F\in\mathcal{O}_d(U)$ such that F is a field of frame of the restriction $D_{|U}$.

3.2. O'_m contains an open dense subset of $\mathcal{D}_m$. Let $d>2n$, $N=2d$, and U be an open subset of M.

Lemma 3.2. *The set $\mathcal{O}_d(U)$ contains a subset $\widetilde{\mathcal{O}_d(U)}$ which is open and dense in $VF_0^m(U)$. Moreover, the complement of $\widetilde{\mathcal{O}_d(U)}$ in $VF_0^m(U)$ is of codimension greater than or equal to $d-n>n$.*

In order to apply transversality techniques, we need to compute the codimension of the typical fiber $T(d,U)$ of $B(d,U)$ in J^N_mTU. For that purpose, we show that $T(d,U)$ is contained in a particular semi-algebraic variety given below.

We may assume that U is the domain of a chart (x,U) of M, centered at $q\in M$.

3.2.1. Construction of semi-algebraic varieties. Let $P(n,N)$ be the set of polynomial mappings $(Q^1,\dots,Q^n):\mathbb{R}^n\to\mathbb{R}^n$ such that $\deg Q^j\leqslant N$, for $1\leqslant j\leqslant n$, and $P(n,N)^m$ be the set of m-tuples $Q=(Q_1,\dots,Q_m)$ with $Q_i\in P(n,N)$, $i=1,\dots,m$. Define the semi-algebraic open subset Ω of $P(n,N)^m$ as the set of $Q\in P(n,N)^m$ such that $Q_1(0),\dots,Q_m(0)$ are linearly independent.

Let $((x,\lambda),\pi^{-1}(U))$ be the induced chart on T^*M. We consider elements of $P(n,N)^m$ as m-tuples of vector fields on U given in local coordinates.

The typical fiber $T_{m,N}$ of the vector bundle $J^N_mTU\times_U T^*U$ is equal to $P(n,N)^m\times\mathbb{R}^n$. The set $\widehat{B}(d)$ is a semi-algebraic subbundle of $J^N_mTM\times_M T^*M$. Its typical fiber $\widehat{T}(d)$ is clearly equal to $\widehat{G}^0(d)\cup\widehat{G}^1(d)$, where $\widehat{G}^0(d)$ and $\widehat{G}^1(d)$ are defined below.

Definition of $\widehat{G}^0(d)$. If $m=2$, set $\widehat{G}^0(d)=\emptyset$. Assume $m\geqslant 3$. For $\sigma\in\mathfrak{S}_m$, $r\leqslant m-3$ an even integer, and $\bar{s}\in N_{p,d}$, $0\leqslant p<m$, define $\phi^0_{\sigma,r,\bar{s}}:T_{m,N}\to\mathbb{R}^d$ as the mapping that associates to $(Q,\lambda)\in T_{m,N}$ the following evaluations of elementary determinants associated to Q:

(i) $\Delta^{r,\dots,r+i}_{0,s_1,\dots,s_{i-1},s}(\sigma,\psi_\lambda)$, for $i=1,\dots,p$ and $s=1,\dots,s_i-1$;
(ii) $\Delta^{r,\dots,r+p,r+p+1}_{0,s_1,\dots,s_p,s}(\sigma,\psi_\lambda)$, for $s=1,\dots,d+p-(s_1+\cdots+s_p)$,

where ψ_λ denotes the element of T^*_qM given in coordinates by $(0,\lambda)$.

Let $T^0_{\sigma,r,\bar{s}}\subset T_{m,N}$ be the open set defined by

$$Q\in\Omega,\quad \Delta^{r,\dots,r+i}_{0,s_1,\dots,s_i}(\sigma,\psi_\lambda)\neq 0,\quad i=0,\dots,p,$$

and let $\widehat{G}^0(d,\sigma,r,\bar{s})$ be the inverse image of $\{0\}$ by the restriction of $\phi^0_{\sigma,r,\bar{s}}$ to $T^0_{\sigma,r,\bar{s}}$. We define $\widehat{G}^0(d)$ as the union of $\widehat{G}^0(d,\sigma,r,\bar{s})$ for $\sigma\in\mathfrak{S}_m$, $r\leqslant m-3$ an even integer, and $\bar{s}\in N_{p,d}$ with $0\leqslant p<m$.

Definition of $\widehat{G}^1(d)$. If m is odd, set $\widehat{G}^1(d)=\emptyset$. Assume that $m\geqslant 2$ is even. For $\sigma\in\mathfrak{S}_m$ and a positive integer $s_1\leqslant d$, define $\phi^1_{\sigma,s_1}:T_{m,N}\to\mathbb{R}^d$ as the mapping that associates to $(Q,\lambda)\in T_{m,N}$ the following evaluations of elementary determinants associated to Q:

(i) $\delta^{m-1}_s(\sigma,\psi_\lambda)$, for $s=0,\dots,s_1-1$;
(ii) $\delta_{s_1,s}(\sigma,\psi_\lambda)$, for $s\in\{1,\dots,d-s_1\}$.

Let $T^1_{\sigma,s_1}\subset T_{m,N}$ be the open set defined by

$$Q\in\Omega,\ \delta^{m-1}_{s_1}(\sigma,\psi_\lambda)\neq 0\quad\text{if } s_1<d,\quad\text{and } \Delta^{m-2}_0(\sigma,\psi_\lambda)\neq 0,$$

and $\widehat{G}^1(d,\sigma,s_1)$ be the inverse image of $\{0\}$ by the restriction of ϕ^1_{σ,s_1} to T^1_{σ,s_1}. We define $\widehat{G}^1(d)$ as the union of $\widehat{G}^1(d,\sigma,s_1)$ for $\sigma\in\mathfrak{S}_m$ and $s_1\leqslant d$.

3.2.2. Evaluation in coordinates of the elementary determinants. We first need to express some of the elementary determinants using the functions h_I's. By an easy but lengthy inductive argument, the next two lemmas follow.

Lemma 3.3. *Assume $m\geqslant 3$. Let $r\leqslant m-3$ be an even integer. With the convention $m+1=r+1$ in the multi-indices I of $\{1,\dots,m\}$, we have:*

- *for $k\in\{r+1,\dots,m\}$ and $s\geqslant 0$,*

$$\Delta^{r,k}_{0,s}=h_{(k+1)k^{s+1}}\big(\Delta^r_0\big)^{s+1}+R^{r,k}_{0,s}, \tag{10}$$

  where $R^{r,k}_{0,s}$ is a polynomial in the h_I's, $|I|\leqslant s+2$, with I different from $(j+1)j^{s+1}$ and $j(j+1)j^s$, for every $j>r$;
- *for p,k such that $r<r+p<k\leqslant m$, $s_1,\dots,s_p\geqslant 1$, and $s\geqslant 0$,*

$$\begin{aligned}&\Delta^{r,r+1,\dots,r+p,k}_{0,s_1,\dots,s_p,s}\\ &\quad=h_{(k+1)k^\ell}\Delta^r_0\left(\prod_{q=0}^{p-1}\left(\Delta^{r,r+1,\dots,r+q}_{0,s_1,\dots,s_q}\right)^{s_{q+1}-1}\right)\left(\Delta^{r,r+1,\dots,r+p}_{0,s_1,\dots,s_p}\right)^s\\ &\qquad+R^{r,r+1,\dots,r+p,k}_{0,s_1,\dots,s_p,s},\end{aligned} \tag{11}$$

where $\ell = s_1+\cdots+s_p+s-p+1$ *and* $R^{r,r+1,\ldots,r+p,k}_{0,s_1,\ldots,s_p,s}$ *is a polynomial in the* h_I*'s,* $|I| \leqslant \ell+1$*, with* I *different from* $(j+1)j^\ell$ *and* $j(j+1)j^{\ell-1}$*, for every* $j > r+p$*.*

Lemma 3.4. *If* $m \geqslant 2$*, then:*

- *For* $s \geqslant 0$*,*

$$\begin{aligned} \delta_s^{m-1} &= -h_{m(m-1)^{s+1}}\Big(P^{m-2}\Big)^{2s+1} + R_s^{m-1}, \\ \delta_s^{m} &= h_{(m-1)m^{s+1}}\Big(P^{m-2}\Big)^{2s+1} + R_s^{m}, \end{aligned} \tag{12}$$

  where R_s^{m-1} *and* R_s^m *are polynomials in the* h_I*'s,* $|I| \leqslant s+2$*,* I *different from the multi-indices* $m(m-1)^{s+1}$*,* $(m-1)m(m-1)^s$*,* $(m-1)m^{s+1}$*, and* $m(m-1)m^s$*.*
- *For* $s_1 \geqslant 1$ *and* $s \geqslant 0$*,*

$$\delta_{s_1,s} = h_{(m-1)m^{s_1+s}}\Big(P^{m-2}\Big)^{2s_1-1}\Big(\delta_{s_1}^{m-1}\Big)^s + R_{s_1,s},$$

  where $R_{s_1,s}$ *is a polynomial in the* h_I*'s,* $|I| \leqslant s_1+s+1$*, with* I *different from* $(m-1)m^{s_1+s}$ *and* $m(m-1)m^{s_1+s-1}$*.*

Remark 13. Equation (12) is the consequence of a property of Pfaffians (see [**7**]), namely

$$P = h_{(m-1)m}P^{m-2} + R_P,$$

where R_P is a polynomial in the h_{ij}'s, with $i<j$ and $ij \neq (m-1)m$.

Coordinate systems. We recall a definition of coordinate systems on Ω (see [**12**]).

Set $\mathcal{A}_0 = \{0\}$. For $k \geqslant 1$, we denote by $\mathcal{A}_k$ the set of k-tuples of ordered integers of $\{1,\ldots,n\}$.

For a homogeneous polynomial $f : \mathbb{R}^n \to \mathbb{R}$ of degree k, and $\eta = (\eta_1,\ldots,\eta_k) \in (\mathbb{R}^n)^k$, the polarization of f along η is the real number $\mathcal{P}f(\eta)$, given by

$$\mathcal{P}f(\eta) = D_{\eta_1}\cdots D_{\eta_k}f,$$

where $D_\xi f$, $\xi \in \mathbb{R}^n$, is the differential of f in the direction ξ.

For $\widehat{Q} \in \Omega$, we complete $\widehat{Q}_1(0),\ldots,\widehat{Q}_m(0)$ in a basis of $\mathbb{R}^n$ with $n-m$ vectors $v_{m+1},\ldots,v_n$. Let $V \subset \Omega$ be a neighborhood of $\widehat{Q}$ such that the mapping $e_V : V \to (\mathbb{R}^n)^n$ defined by

$$\begin{aligned} e_V^i(Q) &= Q_i(0) \quad \text{for } i = 1,\ldots,m, \\ e_V^i(Q) &= v_i \quad \text{for } i = m+1,\ldots,n, \end{aligned}$$

associates to each $Q \in V$ a basis of $\mathbb{R}^n$. To this mapping e_V, is associated a coordinate chart on $V \subset \Omega$, given by

$$\big(X^j_{i,\nu} : j = 1,\ldots,n,\ i = 1,\ldots,m,\ \nu \in \mathcal{A}_k,\ k = 0,\ldots,N\big), \tag{13}$$

where $X^j_{i,\nu}$ is the polarization of the j-th coordinate of the homogeneous part of degree $k=|\nu|$ of Q_i along $\big(e_V^{\nu_1}(Q_1,...,Q_m),...,e_V^{\nu_k}(Q_1,...,Q_m)\big)$.

We next evaluate on T^*_qM the elementary determinants associated to an element Q of V. Consider the chart of $\Omega\times\mathbb{R}^n$ of domain $\widehat{V}=V\times\mathbb{R}^n$.

For $\nu\in\mathcal{A}_k$, set $\nu!=\nu_1!\dots\nu_k!$ and $x^\nu=x_1^{\nu_1}\dots x_k^{\nu_k}$. In coordinates, Q_i, $i=1,\dots,m$, is represented by

$$\frac{\partial}{\partial x_i}+\sum_{1\leqslant k\leqslant N,\ \nu\in\mathcal{A}_k}\frac{x^\nu}{\nu!}X_{i,\nu},$$

where each $X_{i,\nu}=\displaystyle\sum_{j=1}^n X^j_{i,\nu}\frac{\partial}{\partial x_j}$ is a constant vector field.

For $i,k\in\{1,\dots,m\}$, we have $[Q_k,Q_i](0)=Q_{ki}(0)=X_{i,k}-X_{k,i}$. By an easy induction, there holds, for $s>1$,

$$Q_{ki^s}(0)=-X_{k,i^s}+R_{i,k,s},$$

where $R_{i,k,s}$ is a polynomial in the coordinates $X^a_{l,\nu}$, with $1\leqslant a\leqslant n$, $1\leqslant l\leqslant m$, $|\nu|\leqslant s$, and ν is different from j^s, $j\in\{1,\dots,m\}$.

Recall that ψ_λ is given in coordinates by $(0,\lambda)$, $\lambda=(\lambda_b)_{1\leqslant b\leqslant n}$. We infer that $h_{ki}(\psi_\lambda)=\langle\lambda,X_{i,k}\rangle-\langle\lambda,X_{k,i}\rangle$ and, for $s>1$,

$$h_{ki^s}(\psi_\lambda)=\langle\lambda,Q_{ki^s}(0)\rangle=-\langle\lambda,X_{k,i^s}\rangle+R'_{i,k,s},$$

where $R'_{i,k,s}$ is a polynomial in the coordinates λ_b, $X^a_{l,\nu}$, with $1\leqslant a,b\leqslant n$, $1\leqslant l\leqslant m$, $|\nu|\leqslant s$, and ν is different from j^s, $j\in\{1,\dots,m\}$.

By an induction argument, $h_I(\psi_\lambda)$ can be expressed in terms of the coordinates λ_b, $X^a_{l,\nu}$. In local coordinates, Lemmas 3.3 and 3.4 yield the following results.

Lemma 3.5. *Assume $m\geqslant 3$. Let $r\leqslant m-3$ be an even integer. With the convention $m+1=r+1$ in the multi-indices I of $\{1,\dots,m\}$, we have:*

- *For $k\in\{r+1,\dots,m\}$ and $s\geqslant 0$,*

$$\Delta^{r,k}_{0,s}(\sigma,\psi_\lambda)=-\langle\lambda,X_{k+1,k^s}\rangle\big(\Delta^r_0(\sigma,\psi_\lambda)\big)^{s+1}+\widetilde{R}^{r,k}_{0,s},$$

  where $\widetilde{R}^{r,k}_{0,s}$ is a polynomial in the coordinates λ_b, $X^a_{l,\nu}$, with $1\leqslant a,b\leqslant n$, $1\leqslant l\leqslant m$, $|\nu|\leqslant s$, and if (l,ν) is of the form $(i+1,i^s)$, then $i\leqslant r$.

- *For p,k such that $r<r+p<k\leqslant m$, $s_1,\dots,s_p\geqslant 1$ and $s\geqslant 0$,*

$$\begin{aligned}&\Delta^{r,r+1,\dots,r+p,k}_{0,s_1,\dots,s_p,s}(\sigma,\psi_\lambda)\\&=-\langle\lambda,X_{k+1,k^\ell}\rangle\Delta^r_0(\sigma,\psi_\lambda)\left(\prod_{q=0}^{p-1}\left(\Delta^{r,r+1,\dots,r+q}_{0,s_1,\dots,s_q}(\sigma,\psi_\lambda)\right)^{s_{q+1}-1}\right)\\&\quad\cdot\left(\Delta^{r,r+1,\dots,r+p}_{0,s_1,\dots,s_p}(\sigma,\psi_\lambda)\right)^s+\widetilde{R}^{r,r+1,\dots,r+p,k}_{0,s_1,\dots,s_p,s},\end{aligned}$$

where $\ell = s_1 + \cdots + s_p + s - p + 1$ *and* $R^{r,r+1,\ldots,r+p,k}_{0,s_1,\ldots,s_p,s}$ *is a polynomial in the coordinates* λ_b, $X^a_{l,\nu}$, *with* $1 \leqslant a, b \leqslant n$, $1 \leqslant l \leqslant m$, $|\nu| \leqslant \ell$, *and if* (l,ν) *is of the form* $(i+1, i^\ell)$, *then* $i \leqslant r+p$.

Lemma 3.6. *If* $m \geqslant 2$, *then:*

- *For* $s \geqslant 0$,

$$\delta^{m-1}_s(\sigma,\psi_\lambda) = \langle \lambda, X_{m,(m-1)^{s+1}} \rangle \big(P^{m-2}(\sigma,\psi_\lambda)\big)^{2s+1} + \widetilde{R}^{m-1}_s,$$
$$\delta^{m}_s(\sigma,\psi_\lambda) = -\langle \lambda, X_{(m-1),m^{s+1}} \rangle \big(P^{m-2}(\sigma,\psi_\lambda)\big)^{2s+1} + \widetilde{R}^{m}_s,$$

  where $\widetilde{R}^{m-1}_s$ *and* $\widetilde{R}^{m}_s$ *are polynomials in the coordinates* λ_b, $X^a_{l,\nu}$, *with* $1 \leqslant a, b \leqslant n$, $1 \leqslant l \leqslant m$, $|\nu| \leqslant s+1$, *and* (l,ν) *is different from* $(m,(m-1)^{s+1})$ *and* $((m-1),m^{s+1})$.
- *For* $s_1 \geqslant 1$ *and* $s \geqslant 0$,

$$\delta_{s_1,s}(\sigma,\psi_\lambda)$$
$$= \langle \lambda, X_{(m-1),m^{s_1+s}} \rangle \big(P^{m-2}(\sigma,\psi_\lambda)\big)^{2s_1-1} \big(\delta^{m-1}_{s_1}(\sigma,\psi_\lambda)\big)^{s} + \widetilde{R}_{s_1,s},$$

  where $\widetilde{R}_{s_1,s}$ *is a polynomial in the coordinates* λ_b, $X^a_{l,\nu}$, *with* $1 \leqslant a, b \leqslant n$, $1 \leqslant l \leqslant m$, $|\nu| \leqslant s_1 + s$, *and* (l,ν) *is different from* $((m-1),m^{s_1+s})$.

3.2.3. Proof of Lemma 3.2.

Lemma 3.7. *The typical fiber of* $B(d,U)$ *is of codimension greater than or equal to* $d-n$, *that is greater than* n.

Proof. If $m \geqslant 3$, let $\sigma \in \mathfrak{S}_m$, $r \leqslant m-3$ an even integer, and $\bar{s} \in N_{p,d}$, $0 \leqslant p < m$. Then, using Lemma 3.5, the mapping $\phi^0_{\sigma,r,\bar{s}}$ is a submersion on $T^0_{\sigma,r,\bar{s}} \cap \widehat{V}$, for every chart domain $\widehat{V}$ of $\Omega \times \mathbb{R}^n$. It follows readily that $\widehat{G}^0(d)$ is of codimension d. Similarly, if $m \geqslant 2$ is even, then, using Lemma 3.6, the mapping ϕ^1_{σ,s_1} is a submersion on $T^1_{\sigma,s_1} \cap \widehat{V}$, for every chart domain $\widehat{V}$ of $\Omega \times \mathbb{R}^n$, for every $\sigma \in \mathfrak{S}_m$ and all positive integers $s_1 \leqslant d$. Hence $\widehat{G}^1(d)$ is of codimension d. Therefore the typical fiber $\widehat{T}(d) = \widehat{G}^0(d) \cup \widehat{G}^1(d)$ of $\widehat{B}(d,U)$ is of codimension d in $P(n,N)^m \times \mathbb{R}^n$. By projection, the typical fiber of $B(d,U)$ is of codimension greater than or equal to $d-n$, that is greater than n. q.e.d.

Clearly, $\mathcal{O}_d(U)$ contains the open subset of $VF^m_0(U)$ given by

$$\widetilde{\mathcal{O}_d}(U) = \{F \in VF^m_0(U) : j^N F \notin \overline{B(d,U)}\}.$$

Since $B(d,U)$ is a semi-algebraic subbundle of $J^N_m TU$, $\overline{B(d,U)}$ and $B(d,U)$ have the same codimension in $J^N_m TU$, which is greater than or equal to $d-n > n$.

For this codimension, $j^N F \notin \overline{B(d,U)}$ if and only if $j^N F$ is transverse to $\overline{B(d,U)}$. Therefore the set $\widetilde{\mathcal{O}_d}(U)$ satisfies

$$\widetilde{\mathcal{O}_d}(U) = \{F \in VF_0^m(U) \ : \ j^N F \text{ is transverse to } \overline{B(d,U)}\}. \tag{14}$$

Using the transversality theorem for stratified sets of [**18**], we obtain that $\widetilde{\mathcal{O}_d}(U)$ is an open dense subset in $VF_0^m(U)$. Lemma 3.2 is proved.

3.3. Minimal order property in O'_m. Consider an open set $U \subset M$, and two integers $m \geqslant 2$ and $d > 2n$.

Lemma 3.8. *Let $F \in VF_0^m(U)$ be a m-tuple of vector fields, and D_F the distribution on U generated by F. If $F \in \mathcal{O}_d(U)$, then, along every nontrivial abnormal extremal $\psi(.)$ of D_F, there holds* $\operatorname{rank} G(\psi(t)) \geqslant m-2$ *a.e. on* $[0,1]$.

Since $\operatorname{rank} G(\psi(t))$ is even, Lemma 3.8 implies that, if m is odd, then $m-2$ can be replaced by $m-1$ in the previous statement.

Lemma 3.9. *Assume m is even. Let $F \in VF_0^m(U)$ be a m-tuple of vector fields, and D_F the distribution on U generated by F. If $F \in \mathcal{O}_d(U)$, then, along every nontrivial abnormal extremal $\psi(.)$ of D_F, there holds* $\operatorname{rank} \widetilde{G}(\psi(t)) = m-1$ *a.e. on* $[0,1]$.

We will need the following technical lemma.

Lemma 3.10. *Let $I \subset \mathbb{R}$ be a compact interval and $f : I \to \mathbb{R}$ be an absolutely continuous function on I. Then, for every measurable subset $J \subset I$ of positive measure such that $f \equiv 0$ on J, one has $f' = 0$ a.e. on J.*

Proof of Lemma 3.10. Since f is of bounded variation on $[0,1]$, its set of discontinuities S is at most countable. Then clearly $f(x) = 0$ except in $J \cap S$. The argument of [**31**, Lemma p. 177] applies to the present situation. q.e.d.

Proof of Lemma 3.8. Let $F \in \mathcal{O}_d(U)$. We may suppose $m \geqslant 3$. Arguing by contradiction, we assume that there exist a nontrivial abnormal extremal $\psi(.)$ of D_F and a measurable set $J \subset [0,1]$ of positive measure, such that rank $G \leqslant m-3$ on J, where G denotes the Goh matrix along $\psi(.)$.

In what follows, a *p-symmetric minor* of G denotes a determinant of the form $\det(h_{ij})_{(i,j)\in I_p^2}$, where I_p is any subset of $\{1,\dots,m\}$ with cardinality p. Let $0 \leqslant r \leqslant m-3$ be the largest even integer such that a r-symmetric minor of G is not identically equal to zero on J. Then, there exists a permutation $\sigma \in \mathfrak{S}_m$, so that $\Delta_0^r(\sigma, \psi(\cdot))$ does not vanish on some subset $J_r \subset J$ of positive measure (see Section 3.1.1 for a definition of the elementary determinants). Moreover, every $(r+2)$-symmetric minor is identically equal to zero on J_r and the rank of the

corresponding sub-matrix of G is less or equal to r. Therefore, every minor of size $r+1$ extracted from such a matrix is also identically equal to zero on J_r. In particular, it implies that, for $k = r+1, \dots, m$,

$$\Delta^{r,k}_{0,0}(\sigma, \psi(t)) = 0, \ t \in J_r. \tag{15}$$

For the sake of simplicity, in the sequel the index i stands for $\sigma(i)$, and we drop the arguments $(\sigma, \psi(\cdot))$.

Let $u(\cdot) = (u_1(\cdot), \dots, u_m(\cdot))$ be the control associated to $\psi(\cdot)$. Differentiating (15) with respect to t, one gets, by Lemma 3.10,

$$u_1\{\Delta^{r,k}_{0,0}, h_1\} + \cdots + u_m\{\Delta^{r,k}_{0,0}, h_m\} = 0 \quad \text{a.e. on } J_r, \tag{16}$$

for $k = r+1, \dots, m$. Equation (16), together with Equation (5) of Section 2.4, imply that the matrix

$$G_0 = \begin{pmatrix} h_{11} & \cdots & h_{1m} \\ \vdots & & \vdots \\ h_{r1} & \cdots & h_{rm} \\ \{\Delta^{r,r+1}_{0,0}, h_1\} & \cdots & \{\Delta^{r,r+1}_{0,0}, h_m\} \\ \vdots & & \vdots \\ \{\Delta^{r,m}_{0,0}, h_1\} & \cdots & \{\Delta^{r,m}_{0,0}, h_m\} \end{pmatrix}$$

is not invertible on J_r. The first diagonal minor of order r of G_0 is Δ^r_0, which never vanishes on J_r. By definition, the diagonal minors of order $r+1$ containing Δ^r_0 are the $\Delta^{r,k}_{0,1}$, $k = r+1, \dots, m$.

We claim that there exist $k_1 \in \{r+1, \dots, m\}$, an integer s_1 with $1 \leqslant s_1 < d+1$, and a subset $J_{r+1} \subset J_r$ of positive measure such that

- $\Delta^{r,k}_{0,l} \equiv 0$ on J_{r+1}, for $k = r+1, \dots, m$ and $l = 0, \dots, s_1 - 1$;
- Δ^{r,k_1}_{0,s_1} never vanishes on J_{r+1}.

Indeed, assume the claim is false. Then the $\Delta^{r,k}_{0,1}$, $k = r+1, \dots, m$, are identically equal to zero on J_r, and we consider the matrix G_1 obtained by replacing the last $m-r$ rows of G_0 by the rows

$$(\{\Delta^{r,k}_{0,1}, h_l\})_{1 \leqslant l \leqslant m}, \ k = r+1, \dots, m.$$

By construction, $\det G_1 \equiv 0$ on J_r. The contradiction assumption implies that, for $k = r+1, \dots, m$, the minor $\Delta^{r,k}_{0,2}$ of G_1 is identically equal to zero on J_r. Proceeding similarly, we get that there exists $t \in J_r$ such that $j^N_{q(t)}F$ belongs to $\widehat{B}^0(d, \sigma, r, 0, \psi(t))$, which contradicts $F \in \mathcal{O}_d(U)$. The claim is thus proved.

Up to a permutation, we assume $k_1 = r+1$. Define now a non-invertible matrix by replacing in G_0:

- the $(r+1)$-th line by $(\{\Delta^{r,r+1}_{0,s_1-1}, h_l\})_{1 \leqslant l \leqslant m}$;
- for $j = r+2, \dots, m$, the j-th line by $(\{\Delta^{r,j}_{0,s_1-1}, h_l\})_{1 \leqslant l \leqslant m}$.

To this matrix is applied the previous reasoning on G_0. We thus obtain, in a finite number of steps, a subset $J_{m-1} \subset J$ of positive measure, and $\bar{s} = (0, s_1, \ldots, s_{m-1})$ in $N_{m-1,d}$, such that

- $\Delta^{r,\ldots,r+m-1}_{0,s_1,\ldots,s_{m-1}}$ never vanishes on J_{m-1};
- $\Delta^{r,\ldots,r+m-1,r+m}_{0,s_1,\ldots,s_{m-1},l} \equiv 0$ on J_{m-1}, $l \geqslant 0$.

As a consequence, $j^N_{q(t)}F$ belongs to the set $\widehat{B}^0(d, \sigma, r, \bar{s}, \psi(t))$ for every $t \in J_{m-1}$, which contradicts $F \in \mathcal{O}_d(U)$. q.e.d.

Proof of Lemma 3.9. Assume there exist a nontrivial abnormal extremal $\psi(.)$ of D_F and a subset $J \subset [0,1]$ of positive measure such that $\operatorname{rank} \widetilde{G}(\psi(t)) \leqslant m-2$ on J. By the previous proof, we may assume that Δ_0^{m-2} is never vanishing on J; in particular $\operatorname{rank} G(\psi(t)) = m-2$. Moreover, for $k = m-1$ and $k = m$, $\delta_0^k \equiv 0$ and $\delta_1^k \equiv 0$ (see Section 3.1.1 for a definition).

Similarly to the argument of Lemma 3.8, there exist a positive integer $s_1 < d$ and $k_1 \in \{m-1, m\}$, such that $\delta^{k_1}_{s_1}$ is not identically equal to zero on J. Indeed, otherwise, $\delta^k_s \equiv 0$, for $s = 0, \ldots, d$ and $k \in \{m-1, m\}$. In that case, for every $t \in J$, $j^N_{q(t)}F$ belongs to $\widehat{B}^1(d, \sigma, d, \psi(t))$, which contradicts $F \in \mathcal{O}_d(U)$.

Up to a permutation, we assume $k_1 = m-1$. Let $J_1 \subset J$ be a subset of positive measure on which $\delta^{m-1}_{s_1}$ is never vanishing. Similarly to the argument of Lemma 3.8, for every $s \geqslant 0$, we have $\delta_{s_1,s} \equiv 0$ on J_1. Then, for every $t \in J_1$, $j^N_{q(t)}F$ belongs to $\widehat{B}^1(d, \sigma, s_1, \psi(t))$, which contradicts $F \in \mathcal{O}_d(U)$. q.e.d.

3.4. Proof of Proposition 3.1. Let k be an integer greater than n, and $d = k + n$. From Lemma 3.2, the set O'_m contains an open dense subset of $\mathcal{D}_m$ of codimension greater than or equal to k.

Let D be a distribution in O'_m and $\psi(\cdot)$ a nontrivial abnormal extremal of D. For every $t_0 \in [0,1]$, we choose a neighborhood U of $\psi(t_0)$ and a m-tuple F of vector fields such that F is a field of frame of $D_{|U}$ and $F \in \mathcal{O}_d(U)$. There exists a closed subinterval I of $[0,1]$ centered at t_0 so that $\psi(\cdot)_{|I}$ is (up to reparameterization) an abnormal extremal of $D_{|U}$. From Lemma 3.8 (resp. Lemma 3.9), there holds $\operatorname{rk} \omega_D(\psi(t)) = m-1$ if m is odd (resp. $\operatorname{rk} \tilde{\omega}_D(\psi(t)) = m-1$ if m is even), for a.e. $t \in I$.

4. Genericity of the corank one property

Proposition 4.1. *There exists $O_m \subset O'_m$, containing an open dense subset of $\mathcal{D}_m$, such that every nontrivial singular curve of a distribution in O_m is of corank one.*

The argument follows the same lines as the proof of Proposition 3.1. We provide the main steps, omitting the details.

First step. We use $\mathcal{F}$ to denote the set of finite sums of mappings on the fiber product $T^*M \times_M T^*M$ of the form $F_1(\psi^{[1]})F_2(\psi^{[2]})$, where $F_1(\cdot), F_2(\cdot) \in C^\infty(T^*M)$. For $k = 1, 2$, we define the following functions on $\mathfrak{S}_m \times (T^*M \times_M T^*M)$:

- $h_I^{[k]}(\sigma, \psi^{[1]}, \psi^{[2]}) = h_{\sigma(I)}(\psi^{[k]})$, for every multi-index I of $\{1, ..., m\}$,
- $\Delta_0^{[k],r}(\sigma, \psi^{[1]}, \psi^{[2]}) = \Delta_0^r(\sigma, \psi^{[k]})$, for $r < m$,
- $P^{[k]}(\sigma, \psi^{[1]}, \psi^{[2]}) = P(\psi^{[k]})$, and $P^{[k],m-2}(\sigma, \psi^{[1]}, \psi^{[2]}) = P^{m-2}(\psi^{[k]})$,
- $\delta_s^{[k],i}(\sigma, \psi^{[1]}, \psi^{[2]}) = \delta_s^i(\sigma, \psi^{[k]})$, for $s \geqslant 0$.

Notice that the restrictions of these functions to $T^*M \times_M T^*M$ belong to $\mathcal{F}$. For $s \geqslant 0$, we define inductively the following functions on $\mathfrak{S}_m \times (T^*M \times_M T^*M)$:

$$\Theta_{s+1} = \det \begin{pmatrix} \big(h_{ij}^{[1]}\big)_{\substack{1 \leqslant i \leqslant m-1 \\ 1 \leqslant j \leqslant m}} \\ \hline \big(\mathcal{L}_{\overrightarrow{h}_j} \Theta_s\big)_{1 \leqslant j \leqslant m} \end{pmatrix},$$

$$\Theta_0 = \det \begin{pmatrix} \big(h_{ij}^{[1]}\big)_{\substack{1 \leqslant i \leqslant m-1 \\ 1 \leqslant j \leqslant m}} \\ \hline \big(h_{1j}^{[2]}\big)_{1 \leqslant j \leqslant m} \end{pmatrix},$$

and

$$\theta_{s+1} = \det \begin{pmatrix} \big(h_{ij}^{[1]}\big)_{\substack{1 \leqslant i \leqslant m-2 \\ 1 \leqslant j \leqslant m}} \\ \hline \big(\{P^{[1]}, h_j^{[1]}\}\big)_{1 \leqslant j \leqslant m} \\ \hline \big(\mathcal{L}_{\overrightarrow{h}_j} \theta_s\big)_{1 \leqslant j \leqslant m} \end{pmatrix},$$

$$\theta_0 = \det \begin{pmatrix} \big(h_{ij}^{[1]}\big)_{\substack{1 \leqslant i \leqslant m-1 \\ 1 \leqslant j \leqslant m}} \\ \hline \big(\{P^{[1]}, h_j^{[1]}\}\big)_{1 \leqslant j \leqslant m} \\ \hline \big(\{P^{[2]}, h_j^{[2]}\}\big)_{1 \leqslant j \leqslant m} \end{pmatrix},$$

where $\mathcal{L}_{\overrightarrow{h}_j} : \mathcal{F} \to \mathcal{F}$ is defined as follows. For $F_1(\cdot), F_2(\cdot) \in C^\infty(T^*M)$, set

$$\begin{aligned} \mathcal{L}_{\overrightarrow{h}_j} \Big(F_1(\psi^{[1]})F_2(\psi^{[2]})\Big) =& F_2(\psi^{[2]}) L_{\overrightarrow{h}_j} \Big(F_1(\psi^{[1]})\Big) \\ &+ F_1(\psi^{[1]}) L_{\overrightarrow{h}_j} \Big(F_2(\psi^{[2]})\Big), \end{aligned}$$

and extend it by linearity to $\mathcal{F}$.

These determinants can be expanded as follows:

$$\Theta_s = \big(\Delta_0^{[1],m-1}\big)^s \big(\Delta_0^{[1],m-1} h_{1m^s}^{[2]} + d_0 h_{1m^s}^{[1]}\big) + R_s,$$

where R_s is a polynomial in the $h_I^{[k]}$, with $k = 1, 2$, $|I| \leqslant s+1$, I different from $1m^s$ and $m1m^{s-1}$, and d_0 is a polynomial in the $h_J^{[k]}$, with $k = 1, 2$, $|J| = 2$, J different from $1m$ and $m1$;

$$\theta_s = \big(\delta_1^{[1],m-1}\big)^s \big(\delta_1^{[1],m-1} P^{[2],m-2} h^{[2]}_{(m-1)m^{s+2}} + d_0' h^{[1]}_{(m-1)m^{s+2}}\big) + R_s',$$

where R_s' is a polynomial in the $h_I^{[k]}$, with $k = 1, 2$, $|I| \leqslant s+3$, I different from $(m-1)m^{s+2}$ and $m(m-1)m^{s+1}$, and d_0' is a polynomial in the $h_J^{[k]}$, with $k = 1, 2$, $|J| \leqslant 3$, J different from $m(m-1)m$.

Second step. Let $d \in \mathbb{N}$, and $N = d + 1$. As in Lemmas 3.5 and 3.6, we express Θ_s and θ_s in local coordinates $(X^a_{l,\nu})$ of domain $V \subset \Omega$. We use $\psi_\lambda^{[k]}$ to denote the element of T_q^*M given in coordinates by $\lambda^{[k]} = (\lambda_b^{[k]})_{1 \leqslant b \leqslant n}$.

Lemma 4.2. *For $s \geqslant 0$, we have*

$$\begin{aligned}
&\Theta_s(\sigma, \psi_\lambda^{[1]}, \psi_\lambda^{[2]}) \\
&= \big\langle \Delta_0^{m-1}(\sigma, \psi_\lambda^{[1]})\lambda^{[2]} + d_0(\lambda^{[1]}, \lambda^{[2]})\lambda^{[1]}, X_{1,m^s} \big\rangle \big(\Delta_0^{m-1}(\sigma, \psi_\lambda^{[1]})\big)^s + \widetilde{R}_s,
\end{aligned}$$

where d_0 is a polynomial function, and $\widetilde{R}_s$ is a polynomial in the coordinates $\lambda_b^{[k]}$, $X^a_{l,\nu}$, with $k = 1, 2$, $1 \leqslant a, b \leqslant n$, $1 \leqslant l \leqslant m$, $|\nu| \leqslant s$, and (l, ν) different from $(1, m^s)$;

$$\begin{aligned}
&\theta_s(\sigma, \psi_\lambda^{[1]}, \psi_\lambda^{[2]}) \\
&= \big\langle \delta_1^{m-1}(\sigma, \psi_\lambda^{[1]}) P^{m-2}(\sigma, \psi_\lambda^{[2]})\lambda^{[2]} + d_0'(\lambda^{[1]}, \lambda^{[2]})\lambda^{[1]}, X_{(m-1),m^{s+2}} \big\rangle \\
&\qquad \cdot \big(\delta_1^{m-1}(\sigma, \psi_\lambda^{[1]})\big)^s + \widetilde{R}_s'
\end{aligned}$$

where d_0' is a polynomial function, and $\widetilde{R}_s'$ is a polynomial in the coordinates $\lambda_b^{[k]}$, $X^a_{l,\nu}$, with $k = 1, 2$, $1 \leqslant a, b \leqslant n$, $1 \leqslant l \leqslant m$, $|\nu| \leqslant s + 2$, and (l, ν) different from $((m-1), m^{s+2})$.

Remark 14. The functions d_0 and d_0' play no role. Indeed, we only need, when $\lambda^{[1]}$ and $\lambda^{[2]}$ are linearly independent, the coefficient of X_{1,m^s} in Θ_s to be nonzero if $\Delta_0^{m-1}(\sigma, \psi_\lambda^{[1]}) \neq 0$, and the coefficient of $X_{(m-1),m^{s+2}}$ in θ_s to be nonzero if $\delta_1^{m-1}(\sigma, \psi_\lambda^{[1]}) P^{m-2}(\sigma, \psi_\lambda^{[2]}) \neq 0$.

On the domain $\widehat{V} = V \times \mathbb{R}^n \times \mathbb{R}^n$, for every $\sigma \in \mathfrak{S}_m$, we define $\phi_{\sigma,\widehat{V}} : \widehat{V} \to \mathbb{R}^d$ as the mapping that associates to $(Q, \lambda^{[1]}, \lambda^{[2]}) \in \widehat{V}$ the evaluations of either $\Theta_s(\sigma, \psi_\lambda^{[1]}, \psi_\lambda^{[2]})$, $0 \leqslant s \leqslant d-1$, if m is odd, or $\theta_s(\sigma, \psi_\lambda^{[1]}, \psi_\lambda^{[2]})$, $0 \leqslant s \leqslant d-1$, if m is even. Let $\widehat{V}_\sigma$ be the open subset of $\widehat{V}$ defined by

$$\lambda^{[1]}, \lambda^{[2]} \text{ linearly independent, and}$$

$$\begin{cases} \Delta_0^{m-1}(\sigma, \psi_\lambda^{[1]}) \neq 0, & \text{if } m \text{ is odd,} \\ \delta_1^{m-1}(\sigma, \psi_\lambda^{[1]}) P^{m-2}(\sigma, \psi_\lambda^{[2]}) \neq 0, & \text{if } m \text{ is even.} \end{cases}$$

The set $\widehat{G}_C(d, \widehat{V}) = \bigcup_{\sigma \in \mathfrak{S}_m} \phi^{-1}_{\sigma, \widehat{V}}(0) \cap \widehat{V}_\sigma$ is a semi-algebraic variety of $\widehat{V}$ of codimension d (see Remark 14).

Third step. We consider $d > 3n$, and $N = d + 1$. For $\sigma \in \mathfrak{S}_m$, let $\widehat{B}_C(d, \sigma)$ be the subset of $J_m^N TM \times_M T^*M \times_M T^*M$ of all triples $(j_q^N F, \psi^{[1]}, \psi^{[2]})$, $q = \pi(\psi^{[1]}) = \pi(\psi^{[2]})$, such that:

1) $f_1(q), \ldots, f_m(q)$ are linearly independent;
2) $\psi^{[1]}, \psi^{[2]}$ are linearly independent;
3) $\Delta_0^{m-1}(\sigma, \psi^{[1]}) \neq 0$ if m is odd, $\delta_1^{m-1}(\sigma, \psi^{[1]}) P^{m-2}(\sigma, \psi^{[2]}) \neq 0$ if m is even;
4) for $s = 0, \ldots, d-1$,

$$\Theta_s(\sigma, \psi^{[1]}, \psi^{[2]}) = 0 \text{ if } m \text{ is odd}, \quad \theta_s(\sigma, \psi^{[1]}, \psi^{[2]}) = 0 \text{ if } m \text{ is even.}$$

Set $\widehat{B}_C(d) = \bigcup_{\sigma \in \mathfrak{S}_m} \widehat{B}_C(d, \sigma)$, and define the "bad set" $B_C(d)$ as the canonical projection of $\widehat{B}_C(d)$ on $J_m^N TM$. Reasoning as in Section 3.2.3, we obtain that the typical fiber of $B_C(d)$ has codimension greater than or equal to $d - 2n$ in $P(n, N)^m$, and we get the following result.

Lemma 4.3. *Let $U \subset M$ be an open set. The set*

$$\mathcal{O}_d^1(U) = \{F \in VF_0^m(U) \ : \ \forall q \in U, \ j_q^N F \notin B_{Cq}(d)\}$$

contains an open dense subset of $VF_0^m(U)$ whose complement is of codimension greater than or equal to $d - 2n$.

Fourth step. The last step is similar to Section 3.3. Consider an open subset U of M and an integer $d > 3n$.

Lemma 4.4. *Let $F \in VF_0^m(U)$ be a m-tuple of vector fields, and D_F the distribution on U generated by F. If $F \in \mathcal{O}_d(U) \cap \mathcal{O}_d^1(U)$, then every nontrivial singular curve of D_F is of corank one.*

Proof. Let $F \in \mathcal{O}_d(U) \cap \mathcal{O}_d^1(U)$. We argue by contradiction. Assume there exist two abnormal extremal lifts $\psi^{[1]}(\cdot)$ and $\psi^{[2]}(\cdot)$ of the same singular curve $q(\cdot)$ such that, for some $t_0 \in [0, 1]$, $\psi^{[1]}(t_0)$ and $\psi^{[2]}(t_0)$ are linearly independent. By linearity of abnormal extremal lifting, $\psi^{[1]}(\cdot)$ and $\psi^{[2]}(\cdot)$ are linearly independent everywhere on $[0, 1]$.

Case m odd. For $k = 1, 2$, let $G^{[k]}$ denote the Goh matrix $G(\psi^{[k]})$. Since $F \in \mathcal{O}_d(U)$, it follows from the proof of Lemma 3.8 that there exists a $(m-1)$-symmetric minor of $G^{[1]}$ which is not identically equal to zero on $[0, 1]$. Then, up to a permutation, we assume that $\Delta_0^{[1], m-1}$ is never vanishing on an open subinterval $J \subset [0, 1]$.

Let $u(\cdot)$ be the control function associated to the singular curve $q(\cdot)$. For $i = 1, \dots, m$ and $k = 1, 2$, we have

$$\sum_{j=1}^{m} u_j h_{ij}^{[k]} = 0 \quad \text{a.e. on } [0,1].$$

Hence, the matrix

$$\begin{pmatrix} h_{11}^{[1]} & \cdots & h_{1m}^{[1]} \\ \vdots & & \vdots \\ h_{(m-1)1}^{[1]} & \cdots & h_{(m-1)m}^{[1]} \\ h_{11}^{[2]} & \cdots & h_{1m}^{[2]} \end{pmatrix}$$

is not invertible on $[0,1]$, i.e., $\Theta_0 \equiv 0$. Differentiating with respect to t, we get

$$\sum_{j=1}^{m} u_j \, \mathcal{L}_{\overrightarrow{h}_j} \Theta_0 = 0 \quad \text{a.e. on } [0,1].$$

It implies $\Theta_1 \equiv 0$. Proceeding similarly, we get, on the interval J, that $\Delta_0^{[1],m-1} \neq 0$ and $\Theta_0 \equiv \cdots \equiv \Theta_{d-1} \equiv 0$, which contradicts $F \in \mathcal{O}_d^1(U)$.

Case m even. Since $F \in \mathcal{O}_d(U)$, it follows from the proofs of Lemmas 3.8 and 3.9 that, up to a permutation, $P^{[1],m-2}$ and $\delta_1^{[1],m-1}$ are never vanishing on some open subinterval $J \subset [0,1]$.

Let us show that we can suppose $P^{[2],m-2}$ never vanishing on J. For $\alpha \in [0,1]$, consider $\psi^{[\alpha]} = (1-\alpha)\psi^{[1]} + \alpha\psi^{[2]}$. Since $P^{m-2}\big(\psi^{[\alpha]}(\cdot)\big)$ depends continuously on α, it is never vanishing on J for α small enough. Moreover, the set of abnormal extremal lifts being a vector space, $\psi^{[\alpha]}$ is an abnormal extremal lift of the singular curve $q(\cdot)$ which is linearly independent of $\psi^{[1]}$ if $\alpha > 0$. It then suffices to replace $\psi^{[2]}$ by $\psi^{[\alpha]}$, for some $\alpha > 0$ small enough.

Similarly to the case m odd, $\delta_1^{[1],m-1} P^{[2],m-2}$ is never vanishing on J, and $\theta_0 \equiv \cdots \equiv \theta_{d-1} \equiv 0$ on J, which contradicts $F \in \mathcal{O}_d^1(U)$. q.e.d.

Proposition 4.1 is finally obtained by combining Lemmas 4.3 and 4.4.

5. Appendix

In this appendix we provide a proof of Proposition 2.7. We follow the same lines as in Section 4. Let $(\psi^{[n]}, \psi^{[a]}) \in T^*M \times_M T^*M$, and $q = \pi(\psi^{[n]}) = \pi(\psi^{[a]})$. For every multi-index I of $\{1, \dots, m\}$, set

$$h_I^{[n]}(\psi^{[n]}, \psi^{[a]}) = h_I(\psi^{[n]}), \text{ and } \; h_I^{[a]}(\psi^{[n]}, \psi^{[a]}) = h_I(\psi^{[a]}),$$

and define inductively the following functions in $\mathcal{F}$, depending on the couple $(\psi^{[n]}, \psi^{[a]})$:

$$\beta_{i,0} = h_i^{[a]},$$
$$\beta_{i,s+1} = \sum_{j=1}^{m} h_j^{[n]} \mathcal{L}_{\overrightarrow{h_j}} \beta_{i,s}, \quad s \in \mathbb{N},$$

where $\mathcal{F}$ and $\mathcal{L}_{\overrightarrow{h_j}}$ are defined in Section 4. For $i \in \{1, \dots, m\}$ and $s \geqslant 0$, one has

$$\beta_{i,s} = (h_i^{[n]})^s h^{[a]}_{(i+1)i^s} + R_{i,s},$$

where $R_{i,s}$ is a polynomial in $h_J^{[n]}$ and $h_I^{[a]}$, $|J| \leqslant s$, $|I| \leqslant s+1$, with I different from $(i+1)i^s$ and $i(i+1)i^{s-1}$ (with the convention that the index $m+1$ stands for 1).

Let $d > 3n$ be an integer, and $N = d+1$. For every $i \in \{1, \dots, m\}$, we define $\widehat{B}(d, i, \psi^{[n]}, \psi^{[a]})$ as the set of $j_q^N F \in J^N_{m,q}$ such that the following conditions hold:

1) $f_1(q), \dots, f_m(q)$ are linearly independent;
2) $h_i^{[n]} \neq 0$;
3) $\beta_{i,s} = 0$ for every $s \in \{0, \dots, d-1\}$.

Let $\widehat{B}(d, \psi^{[n]}, \psi^{[a]}) \subset J^N_{m,q}$ be the union of the sets $\widehat{B}(d, i, \psi^{[n]}, \psi^{[a]})$ with $i \in \{1, \dots, m\}$. Define now $\widehat{B}(d) \subset J^N_m TM \times_M T^*M \times_M T^*M$ by

$$\widehat{B}(d) = \{(j_q^N F, \psi^{[n]}, \psi^{[a]})) : j_q^N F \in \widehat{B}(d, \psi^{[n]}, \psi^{[a]})\}.$$

Finally, the "bad set" $B(d)$ is the canonical projection of $\widehat{B}(d)$ on $J^N_m TM$.

Reasoning as in Section 3.2.3, it is clear that $\widehat{B}(d)$ is a semi-algebraic subbundle of $J^N_m TM \times_M T^*M \times_M T^*M$ whose typical fiber has codimension greater than or equal to d in $P(n,N)^m \times \mathbb{R}^n \times \mathbb{R}^n$. By projection we deduce that the typical fiber of $B(d)$ has codimension greater than or equal to $d - 2n$, that is greater than n. We get the following result, analogous to Lemma 3.2.

Lemma 5.1. *Let $U \subset M$ be an open set. Then*

$$\mathcal{O}_d^s(U) = \{F \in VF_0^m(U) : \forall q \in U,\ j_q^N F \notin B_q(d)\}$$

contains an open dense subset of $VF_0^m(U)$.

As in Section 3.3, we have the following lemma.

Lemma 5.2. *Let $F \in VF_0^m(U)$ be a m-tuple of vector fields, D_F be the distribution on U generated by F, and g_F be the Riemannian metric on D_F for which F is orthonormal. If $F \in \mathcal{O}_d^s(U)$, then every nontrivial singular curve of D_F is strictly abnormal for the metric g_F.*

Proof. Let $F \in \mathcal{O}_d^s(U)$. By contradiction, assume that there exists a nontrivial singular curve $q(\cdot)$, associated to a control $u(\cdot) = (u_1(\cdot), \dots, u_m(\cdot))$ on $[0,1]$, and having on the one part a normal extremal lift $\psi^{[n]}(\cdot)$ and on the other part an abnormal extremal lift $\psi^{[a]}(\cdot)$. From Remark 12, there holds

$$u_i(t) = h_i(\psi^{[n]}(t)) = h_i^{[n]}(\psi^{[n]}(t), \psi^{[a]}(t)), \tag{17}$$

and

$$h_i^{[a]}(\psi^{[n]}(t), \psi^{[a]}(t)) = h_i(\psi^{[a]}(t)) = 0, \tag{18}$$

for every $i \in \{1, \dots, m\}$, and $t \in [0,1]$. Since the control $u(\cdot)$ is nontrivial, there exist $i_0 \in \{1, \dots, m\}$ and an open interval $J \subset [0,1]$ on which $u_{i_0}(\cdot)$ is never vanishing. Differentiating (18) with respect to t, one gets,

$$\begin{aligned}
0 &= \frac{d}{dt} h_{i_0+1}^{[a]}(\psi^{[n]}(t), \psi^{[a]}(t)) \\
&= \sum_{j=1}^{m} u_j(t) h_{(i_0+1)j}^{[a]}(\psi^{[n]}(t), \psi^{[a]}(t)) \\
&= \sum_{j=1}^{m} h_j^{[n]}(\psi^{[n]}(t), \psi^{[a]}(t)) h_{(i_0+1)j}^{[a]}(\psi^{[n]}(t), \psi^{[a]}(t)) \\
&= \beta_{i_0,1}(\psi^{[n]}(t), \psi^{[a]}(t)),
\end{aligned}$$

for every $t \in J$. By induction,

$$\beta_{i_0,s}(\psi^{[n]}(t), \psi^{[a]}(t)) = 0,$$

for every $s \in \{0, \dots, d-1\}$ and $t \in J$. Hence $j_{q(t)}^N F$ belongs to $\widehat{B}(d, i_0, \psi^{[n]}(t), \psi^{[a]}(t))$ for $t \in J$, which contradicts the hypothesis.

q.e.d.

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